

# Umbrella Antennas for LF

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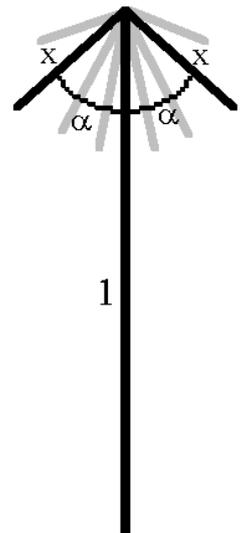
After reading the excellent web page of [ON7YD](#), where the essential formulae are given, I realised that I had enough information to calculate the best shape for an Umbrella Antenna. This is a vertical with top loading wires, which descend to a certain fraction of the height. The question is just what that fraction should be for greatest Radiation Resistance.

It is assumed that the antenna is small compared with a wavelength, which will usually be the case for an Amateur antenna. This means that on any continuous segment of the antenna, the voltage will be approximately constant and that the current will decrease linearly. The rate of decrease depends upon the voltage and the capacity to ground of the segment.

We take the height of the vertical as unity and the length of each rib as  $x$ , i.e. as a proportion of the vertical height. It is supposed that there are  $n$  ribs all of the same length and all inclined at an angle  $\alpha$  to the vertical.

The antenna current is taken as unity at the base and decreases approximately linearly to a value  $i$  at the top of the vertical. With no ribs,  $i$  is zero. The current then divides equally between the ribs and decreases linearly to zero at the end of each rib.

Increasing the length of the ribs increases the capacity of the top hat and therefore increases the value of  $i$  and as a consequence increases the Radiation Resistance,  $R_r$ . However the current down the ribs has a downward vertical component which decreases  $R_r$ . As the length  $x$  of the ribs is increased from zero,  $R_r$  first starts to increase. As the length increases  $R_r$  reaches a maximum value and thereafter decreases. The purpose of the analysis below is to determine the optimum value of  $x$  which produces the greatest value of  $R_r$  for a given number of ribs,  $n$ , and a given angle,  $\alpha$ . We will then explore the effects of increasing  $n$  and changing  $\alpha$ .



If  $\alpha$  were  $180^\circ$ , one rib would constitute an extension of the vertical and the value of  $i$  can be calculated remembering that the current decreases linearly from 1 to zero along the entire length of the antenna. In this case we would have:

$$i = \frac{x}{x+1}$$

When the ribs are horizontal or downward sloping, they contribute more to the top hat capacity than the single upward extension of length  $x$  provided that they are well separated. (This of course will not be true at the centre, but we will come to that later.)

From the formulae for the capacity of horizontal and vertical wires, it has been shown that for typical amateur antennas, 1 metre of horizontal wire contributes approximately 5 pF per metre and 1 metre of vertical wire contributes approximately 6 pF per metre. These figures are practically the same and therefore for  $n$  ribs we can deduce that the value of  $i$  for practical purposes is given by:

$$i = \frac{x}{x + \frac{1}{n}} \quad (1)$$

A more detailed analysis will be given later.

Each antenna segment contributes a component of the Radiation Resistance in proportion to the square of the product of the average current flowing in the segment and the length of the segment.

If we take the Radiation Resistance of a vertical monopole as  $M$ , the Radiation Resistance of the umbrella antenna can be written:

$R_r = p^2 \cdot M$  and we can calculate  $p$  to make this a maximum.

We consider the antenna as consisting of 1 vertical element and  $n$  ribs.

The average current in the vertical element is  $\frac{1+i}{2}$ , and since the average current flowing in a monopole is  $\frac{1}{2}$ , the ratio is simply  $1+i$ .

Similarly the average current flowing in all of the  $n$  ribs taken together is  $\frac{i}{2}$  and as a ratio to the monopole this is simply  $i$ . But this current is downward and has a vertical component  $i \cos(\alpha)$ . Remembering that the length of the vertical is 1 and the length of each rib is  $x$ , it follows that:

$$\begin{aligned}
 p &= 1+i - ix \cos(\alpha) \\
 &= 1+i(1-xc) \\
 &= 1 + \frac{x(1-xc)}{x + \frac{1}{n}} \\
 \text{i.e. } p &= 1 + \frac{x(1-xc)}{x + \frac{1}{n}} \quad (2)
 \end{aligned}$$

To find the value of  $x$  for which this has a maximum, we differentiate with respect to  $x$  and solve for the derivative = 0.

$$\frac{\partial p}{\partial x} = \frac{1-xc}{x + \frac{1}{n}} - \frac{cx}{x + \frac{1}{n}} - \frac{x(1-xc)}{(x + \frac{1}{n})^2}$$

and this is zero when:

$$(1-2cx)(x + \frac{1}{n}) = x(1-cx)$$

$$\therefore cx^2 + \frac{2c}{n}x = \frac{1}{n}$$

$$\therefore x^2 + \frac{2}{n}x = \frac{1}{nc}$$

$$\therefore \left(x + \frac{1}{n}\right)^2 = \frac{1}{n^2} + \frac{1}{nc}$$

$$= \frac{c+n}{n^2c}$$

$$\therefore x = -\frac{1}{n} + \sqrt{\frac{c+n}{n^2c}}$$

$$\text{or } x = \frac{1}{n} \left( \sqrt{1+n \sec(\alpha)} - 1 \right)$$

If we now substitute this value of  $x$  in equation (2), we obtain a maximum value for  $p$ :

$$\begin{aligned}
p &= 1 + \frac{\sqrt{\dots} - 1}{\sqrt{\dots}} \left( 1 - \frac{(\sqrt{\dots} - 1)c}{n} \right) \\
&= 1 + \frac{\sqrt{\dots} - 1}{\sqrt{\dots}} \left( \frac{n/c + 1 - \sqrt{\dots}}{n/c} \right) \\
&= 1 + \frac{c}{n} (\sqrt{\dots} - 1) (\sqrt{\dots} - 1) \\
&= 1 + \frac{c}{n} \left\{ \frac{n}{c} + 2 - 2\sqrt{\dots} \right\} \\
&= 2 + 2\frac{c}{n} \left( 1 - \sqrt{1 + \frac{n}{c}} \right) \\
&= 2\frac{c}{n} \left\{ 1 + \frac{n}{c} - \sqrt{1 + \frac{n}{c}} \right\}
\end{aligned}$$

i.e. the maximum value for p is:

$$p = 2 \left( 1 + \frac{\cos(\alpha)}{n} \right) \left\{ 1 - \sqrt{\frac{\cos(\alpha)}{n + \cos(\alpha)}} \right\}$$

and the value of the Radiation Resistance is

$$R_r = 4 \left( 1 + \frac{\cos(\alpha)}{n} \right)^2 \left\{ 1 - \sqrt{\frac{\cos(\alpha)}{n + \cos(\alpha)}} \right\}^2 M$$

I have tabulated these results for some values of n and alpha using Microsoft Excel and include these results below:

### **Table of values for x**

Values of alpha from 0 to 89° horizontally and Values of n vertically

	0	10	20	30	40	45	50	60	70	80	89
1	0.41	0.42	0.44	0.47	0.52	0.55	0.60	0.73	0.98	1.60	6.64
2	0.37	0.37	0.38	0.41	0.45	0.48	0.51	0.62	0.81	1.27	4.88
3	0.33	0.34	0.35	0.37	0.41	0.43	0.46	0.55	0.71	1.09	4.05
4	0.31	0.31	0.32	0.34	0.37	0.40	0.42	0.50	0.64	0.98	3.54
5	0.29	0.29	0.30	0.32	0.35	0.37	0.39	0.46	0.59	0.89	3.19
6	0.27	0.28	0.29	0.30	0.33	0.35	0.37	0.43	0.55	0.83	2.93
7	0.26	0.26	0.27	0.29	0.31	0.33	0.35	0.41	0.52	0.78	2.72
8	0.25	0.25	0.26	0.27	0.30	0.31	0.33	0.39	0.49	0.73	2.55
9	0.24	0.24	0.25	0.26	0.29	0.30	0.32	0.37	0.47	0.70	2.41
10	0.23	0.23	0.24	0.25	0.27	0.29	0.31	0.36	0.45	0.67	2.30
100	0.09	0.09	0.09	0.10	0.10	0.11	0.12	0.13	0.16	0.23	0.75

Note that there is no maximum for alpha = 90°, the longer the radials the better, short of a quarter wavelength.

It can be seen that the greater the angle, the longer the ribs can be and the more ribs the shorter they must be.

### **Table of values for Rr/M**

Values of alpha from 0 to 89° horizontally and Values of n vertically.

	0	10	20	30	40	45	50	60	70	80	89
1	1.37	1.38	1.39	1.42	1.45	1.48	1.51	1.61	1.77	2.09	3.13
2	1.61	1.61	1.63	1.67	1.72	1.75	1.79	1.91	2.09	2.43	3.35
3	1.78	1.78	1.81	1.84	1.90	1.94	1.98	2.11	2.30	2.63	3.45
4	1.91	1.92	1.94	1.98	2.04	2.08	2.12	2.25	2.44	2.76	3.52
5	2.02	2.02	2.05	2.09	2.15	2.19	2.24	2.36	2.55	2.86	3.57
6	2.11	2.11	2.14	2.18	2.24	2.28	2.33	2.45	2.63	2.93	3.60
7	2.18	2.19	2.21	2.26	2.32	2.36	2.40	2.53	2.71	3.00	3.63
8	2.25	2.26	2.28	2.32	2.38	2.42	2.47	2.59	2.77	3.05	3.65
9	2.31	2.32	2.34	2.38	2.44	2.48	2.53	2.65	2.82	3.09	3.67
10	2.36	2.37	2.39	2.43	2.49	2.53	2.58	2.70	2.86	3.13	3.69
100	3.31	3.31	3.33	3.35	3.38	3.41	3.43	3.49	3.57	3.69	3.90

Note that the maximum that can be achieved is a factor of 4 (6 dB) better than a Monopole.

One can either use greater angles or more ribs to approach the theoretical maximum value of Rr.

### Assumptions

The above formulae were obtained using formula (1) which assumed that the capacity per unit length of the ribs was the same as that for the vertical element. More accurately we may proceed as follows:

The capacity in picofarads per unit length of a horizontal wire at a height of  $h$  metres is given by the formula:

$$C_H = 24 / \log \left( \frac{4h}{d} \right) \text{ where } d \text{ is the diameter of the wire in metres}$$

and the formula for the capacity in picofarads per unit length of a vertical wire at a height  $h$  metres is:

$$C_V = 24 / \log \left( \frac{1.15h}{d} \right)$$

If we change the formulae to use the natural logarithm rather than logs to the base 10, these formulae become:

$$C_H = 55.26 / \ln \left( \frac{4h}{d} \right) \quad (3)$$

and

$$C_V = 55.26 / \ln \left( \frac{1.15h}{d} \right) \quad (4)$$

Therefore for a rib of the umbrella we may calculate the capacity as

$$C_r = \int_0^x \frac{55.26 \sin(\alpha)}{\ln \left( \frac{4(1-l \cos(\alpha))}{d/h} \right)} + \frac{55.26 \cos(\alpha)}{\ln \left( \frac{1.15(1-l \cos(\alpha))}{d/h} \right)} dl$$

Here  $h$  is the actual height of the vertical in metres

$$= 55.26 D \left\{ \frac{1}{4} \tan(\alpha) [Li(z)]_{4^{1-x \cos(\alpha)}}^{4/D} + \frac{1}{1.15} [Li(z)]_{1.15^{1-x \cos(\alpha)}}^{1.15/D} \right\}$$

where  $D$  stands for  $d/h$ .

The function  $Li(z)$  is the Logarithmic Integral and is tabulated. e.g. Jahnke und Emde, Tables of Functions. Calculations with typical values of  $x$ ,  $h$ ,  $d$  and  $\alpha$  show that  $C_r$  does indeed take values of the order of 5 or 6 pF per metre, confirming the approximate correctness of the formulae. Calculations with small values of  $\alpha$  tend to give larger values for  $C_r$ , but I believe that these are invalidated by the closeness of the ribs to the vertical and to one another.