Consider a lossless line, length $\ell$, terminated with a load $Z_L$.

Let's determine the input impedance of this line!

**Q:** Just what do you mean by **input impedance**?

**A:** The input impedance is simply the line impedance seen at the beginning ($z = -\ell$) of the transmission line, i.e.:

$$Z_{\text{in}} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note $Z_{\text{in}}$ equal to neither the load impedance $Z_L$ nor the characteristic impedance $Z_0$!

$$Z_m \neq Z_L \quad \text{and} \quad Z_m \neq Z_0$$
To determine exactly what \( Z_{in} \) is, we first must determine the voltage and current at the \textbf{beginning} of the transmission line \((z = -\ell)\).

\[
V(\z = -\ell) = V_0^+ \left[ e^{+j\beta \ell} + \Gamma_L e^{-j\beta \ell} \right]
\]

\[
I(\z = -\ell) = \frac{V_0^+}{Z_0} \left[ e^{+j\beta \ell} - \Gamma_L e^{-j\beta \ell} \right]
\]

Therefore:

\[
Z_{in} = \frac{V(\z = -\ell)}{I(\z = -\ell)} = Z_0 \left( \frac{e^{+j\beta \ell} + \Gamma_L e^{-j\beta \ell}}{e^{+j\beta \ell} - \Gamma_L e^{-j\beta \ell}} \right)
\]

We can explicitly write \( Z_{in} \) in terms of load \( Z_L \) using the previously determined relationship:

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

Combining these two expressions, we get:

\[
Z_{in} = Z_0 \left( \frac{Z_L + Z_0}{Z_L + Z_0} e^{+j\beta \ell} + \frac{(Z_L - Z_0)}{(Z_L + Z_0)} e^{-j\beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_L (e^{+j\beta \ell} + e^{-j\beta \ell}) + Z_0 \left( e^{+j\beta \ell} - e^{-j\beta \ell} \right)}{Z_L (e^{+j\beta \ell} + e^{-j\beta \ell}) - Z_0 \left( e^{+j\beta \ell} - e^{-j\beta \ell} \right)} \right)
\]

Now, recall \textbf{Euler's equations}:

\[
e^{+j\beta \ell} = \cos \beta \ell + j \sin \beta \ell
\]

\[
e^{-j\beta \ell} = \cos \beta \ell - j \sin \beta \ell
\]
Using Euler’s relationships, we can likewise write the input impedance without the complex exponentials:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right) \\
= Z_0 \left( \frac{Z_L + j Z_0 \tan \beta \ell}{Z_0 + j Z_L \tan \beta \ell} \right)
\]

Note that depending on the values of \( \beta, Z_0 \) and \( \ell \), the input impedance can be radically different from the load impedance \( Z_L \)!

**Special Cases**

Now let’s look at the \( Z_{in} \) for some important load impedances and line lengths.

➔ You should commit these results to memory!

1. \( \ell = \frac{\lambda}{2} \)

If the length of the transmission line is exactly one-half wavelength \( (\ell = \frac{\lambda}{2}) \), we find that:

\[
\beta \ell = \frac{2\pi \lambda}{\lambda/2} = \pi
\]

meaning that:

\[
\cos \beta \ell = \cos \pi = -1 \quad \text{and} \quad \sin \beta \ell = \sin \pi = 0
\]
and therefore:

\[
\begin{align*}
Z_{in} &= Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right) \\
&= Z_0 \left( \frac{Z_L (-1) + j Z_L (0)}{Z_0 (-1) + j Z_L (0)} \right) \\
&= Z_L
\end{align*}
\]

In other words, if the transmission line is precisely one-half wavelength long, the input impedance is equal to the load impedance, regardless of \(Z_0\) or \(\beta\).

If the length of the transmission line is exactly \(\frac{\lambda}{4}\), we find that:

\[
\beta \ell = \frac{2\pi \lambda}{4} = \frac{\pi}{2}
\]

meaning that:

\[
\cos \beta \ell = \cos \frac{\pi}{2} = 0 \quad \text{and} \quad \sin \beta \ell = \sin \frac{\pi}{2} = 1
\]
and therefore:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_L \left(0\right) + j Z_0 \left(1\right)}{Z_0 \left(0\right) + j Z_L \left(1\right)} \right)
\]

\[
= \left(\frac{Z_0}{Z_L}\right)^2
\]

In other words, if the transmission line is precisely one-quarter wavelength long, the input impedance is inversely proportional to the load impedance.

Think about what this means! Say the load impedance is a short circuit, such that \(Z_L = 0\). The input impedance at beginning of the \(\lambda/4\) transmission line is therefore:

\[
Z_{in} = \frac{(Z_0)^2}{Z_L} = \frac{(Z_0)^2}{0} = \infty
\]

\(Z_{in} = \infty\)! This is an open circuit! The quarter-wave transmission line transforms a short-circuit into an open-circuit—and vice versa!
3. \( Z_L = Z_0 \)

If the load is **numerically equal** to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_0 \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_0 \sin \beta \ell} \right)
\]

\[= Z_0 \]

In other words, if the **load** impedance is equal to the transmission line **characteristic** impedance, the **input** impedance will be likewise be equal to \( Z_0 \) regardless of the transmission line length \( \ell \).

4. \( Z_L = j X_L \)

If the load is **purely reactive** (i.e., the resistive component is zero), the input impedance is:
In other words, if the load is purely reactive, then the input impedance will likewise be purely reactive, regardless of the line length $\ell$.

Note that the opposite is not true: even if the load is purely resistive ($Z_L = R$), the input impedance will be complex (both resistive and reactive components).

Q: Why is this?

A:
5. $\ell \ll \lambda$

If the transmission line is electrically small—its length $\ell$ is small with respect to signal wavelength $\lambda$—we find that:

$$\beta \ell = \frac{2\pi}{\lambda} \ell = \frac{2\pi \ell}{\lambda} \approx 0$$

and thus:

$$\cos \beta \ell = \cos 0 = 1 \quad \text{and} \quad \sin \beta \ell = \sin 0 = 0$$

so that the input impedance is:

$$Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)$$

$$= Z_0 \left( \frac{Z_L \left(1\right) + j Z_L \left(0\right)}{Z_0 \left(1\right) + j Z_L \left(0\right)} \right)$$

$$= Z_L$$

In other words, if the transmission line length is much smaller than a wavelength, the input impedance $Z_{in}$ will always be equal to the load impedance $Z_L$.

This is the assumption we used in all previous circuits courses (e.g., EECS 211, 212, 312, 412)! In those courses, we assumed that the signal frequency $\omega$ is relatively low, such that the signal wavelength $\lambda$ is very large ($\lambda \gg \ell$).
Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the same!

\[ V(z = -\ell) \approx V(z = 0) \quad \text{and} \quad I(z = -\ell) \approx I(z = 0) \quad \text{if} \quad \ell \ll \lambda \]

If \( \ell \ll \lambda \), our "wire" behaves exactly as it did in EECS 211!