Antenna Arrays

Antennas with a given radiation pattern may be arranged in a pattern (line, circle, plane, etc.) to yield a different radiation pattern.

Antenna array - a configuration of multiple antennas (elements) arranged to achieve a given radiation pattern.

Linear array - antenna elements arranged along a straight line.

Circular array - antenna elements arranged around a circular ring.

Planar array - antenna elements arranged over some planar surface (example - rectangular array).

Conformal array - antenna elements arranged to conform to some non-planar surface (such as an aircraft skin).

There are several array design variables which can be changed to achieve the overall array pattern design.

<u>Array Design Variables</u>

1. General array shape (linear, circular, planar, etc.).
2. Element spacing.
3. Element excitation amplitude.
4. Element excitation phase.
5. Patterns of array elements.

Phased array - an array of identical elements which achieves a given pattern through the control of the element excitation phasing. Phased arrays can be used to steer the main beam of the antenna without physically moving the antenna.

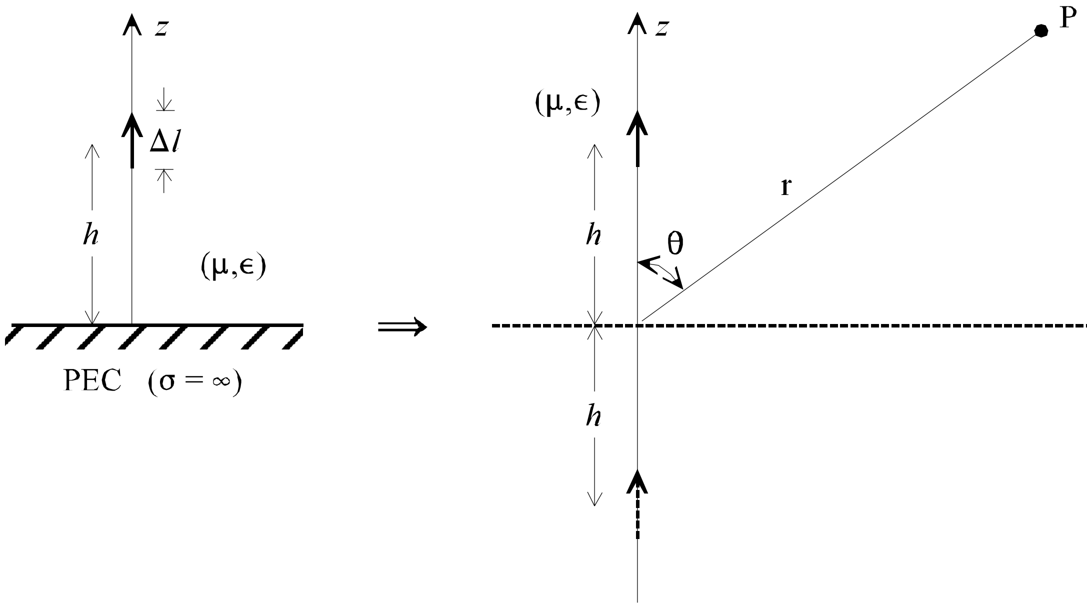
Given an antenna array of identical elements, the radiation pattern of the antenna array may be found according to the *pattern multiplication theorem*.

<u>Pattern multiplication theorem</u>

$$\textit{Array pattern} = \textit{Array element pattern} \times \textit{Array factor}\ (\mathrm{AF})$$

Array element pattern - the pattern of the individual array element.
Array factor - a function dependent only on the geometry of the array and the excitation (amplitude, phase) of the elements.

<u>Example</u> (Pattern multiplication - infinitesimal dipole over ground)



The far field of this two element array was found using image theory to be

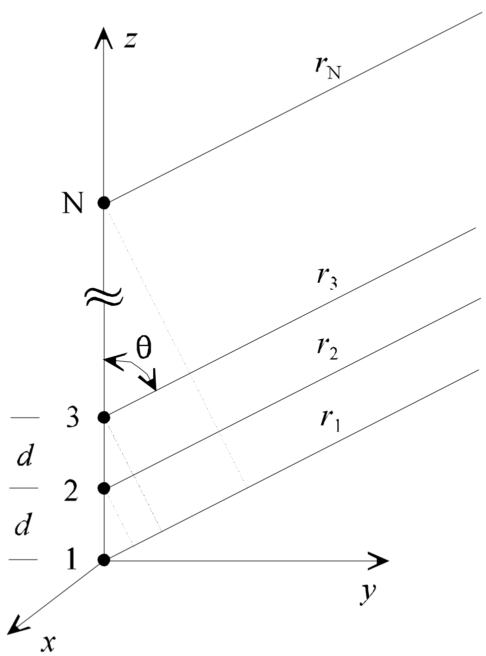
$$E_\theta \approx j\eta \underbrace{\frac{kI_o \Delta l\, e^{-\varphi\kappa\rho}}{4\pi r} \sin\theta}_{\text{element pattern}} \underbrace{\left[\, 2\cos(kh\cos\theta)\,\right]}_{\text{array factor}}$$

N-Element Linear Array

The array factor *AF* is independent of the antenna type assuming all of the elements are identical. Thus, isotropic radiators may be utilized in the derivation of the array factor to simplify the algebra. The field of an isotropic radiator located at the origin may be written as (assuming θ-polarization)

$$E_\theta = I_o \frac{e^{-jkr}}{4\pi r}$$

We assume that the elements of the array are uniformly-spaced with a separation distance *d*.



In the far field of the array

$$r_1 = r$$

$$r_2 \approx r - d\cos\theta$$

$$r_3 \approx r - 2d\cos\theta$$

$$\vdots$$

$$r_N \approx r - (N-1)d\cos\theta$$

The current magnitudes the array elements are assumed to be equal and the current on the array element located at the origin is used as the phase reference (zero phase).

$$I_1 = I_o \qquad I_2 = I_o e^{j\phi_2} \qquad I_3 = I_o e^{j\phi_3} \quad \cdots \quad I_N = I_o e^{j\phi_N}$$

The far fields of the individual array elements are

$$E_{\theta 1} \approx I_o \frac{e^{-jkr}}{4\pi r} = E_o$$

$$E_{\theta 2} \approx I_o e^{j\phi_2} \frac{e^{-jk(r - d\cos\theta)}}{4\pi r} = E_o e^{j(\phi_2 + kd\cos\theta)}$$

$$E_{\theta 3} \approx I_o e^{j\phi_3} \frac{e^{-jk(r - 2d\cos\theta)}}{4\pi r} = E_o e^{j(\phi_3 + 2kd\cos\theta)}$$

$$\vdots$$

$$E_{\theta N} \approx I_o e^{j\phi_N} \frac{e^{-jk[r - (N-1)d\cos\theta]}}{4\pi r} = E_o e^{j[\phi_N + (N-1)kd\cos\theta]}$$

The overall array far field is found using superposition.

$$\begin{aligned} E_\theta &= E_{\theta 1} + E_{\theta 2} + E_{\theta 3} + \cdots + E_{\theta N} \\ &= E_o \left[1 + e^{j(\phi_2 + kd\cos\theta)} + e^{j(\phi_3 + 2kd\cos\theta)} + \cdots + e^{j[\phi_N + (N-1)kd\cos\theta]} \right] \\ &= E_o [AF] \end{aligned}$$

$$\boxed{AF = \left[1 + e^{j(\phi_2 + kd\cos\theta)} + e^{j(\phi_3 + 2kd\cos\theta)} + \cdots + e^{j[\phi_N + (N-1)kd\cos\theta]} \right]}$$

(Array factor for a uniformly-spaced *N*-element linear array)

Uniform N-Element Linear Array

(uniform spacing, uniform amplitude, linear phase progression)

A *uniform array* is defined by uniformly-spaced identical elements of equal magnitude with a linearly progressive phase from element to element.

$$\phi_1 = 0 \qquad \phi_2 = \alpha \qquad \phi_3 = 2\alpha \qquad \cdots \qquad \phi_N = (N-1)\alpha$$

Inserting this linear phase progression into the formula for the general *N*-element array gives

$$\begin{aligned} AF &= \left[1 + e^{j(\alpha + kd\cos\theta)} + e^{j2(\alpha + kd\cos\theta)} + \dots + e^{j(N-1)[\alpha + kd\cos\theta]}\right] \\ &= \left[1 + e^{j\psi} + e^{j2\psi} + \dots + e^{j(N-1)\psi}\right] \qquad (\psi = \alpha + kd\cos\theta) \\ &= \sum_{n=1}^{N} e^{j(n-1)\psi} \end{aligned}$$

The function ψ is defined as the *array phase function* and is a function of the element spacing, phase shift, frequency and elevation angle. If the array factor is multiplied by $e^{j\psi}$, the result is

$$(AF)e^{j\psi} = \left[e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \dots + e^{jN\psi}\right]$$

Subtracting the array factor from the equation above gives

$$AF(e^{j\psi} - 1) = (e^{jN\psi} - 1)$$

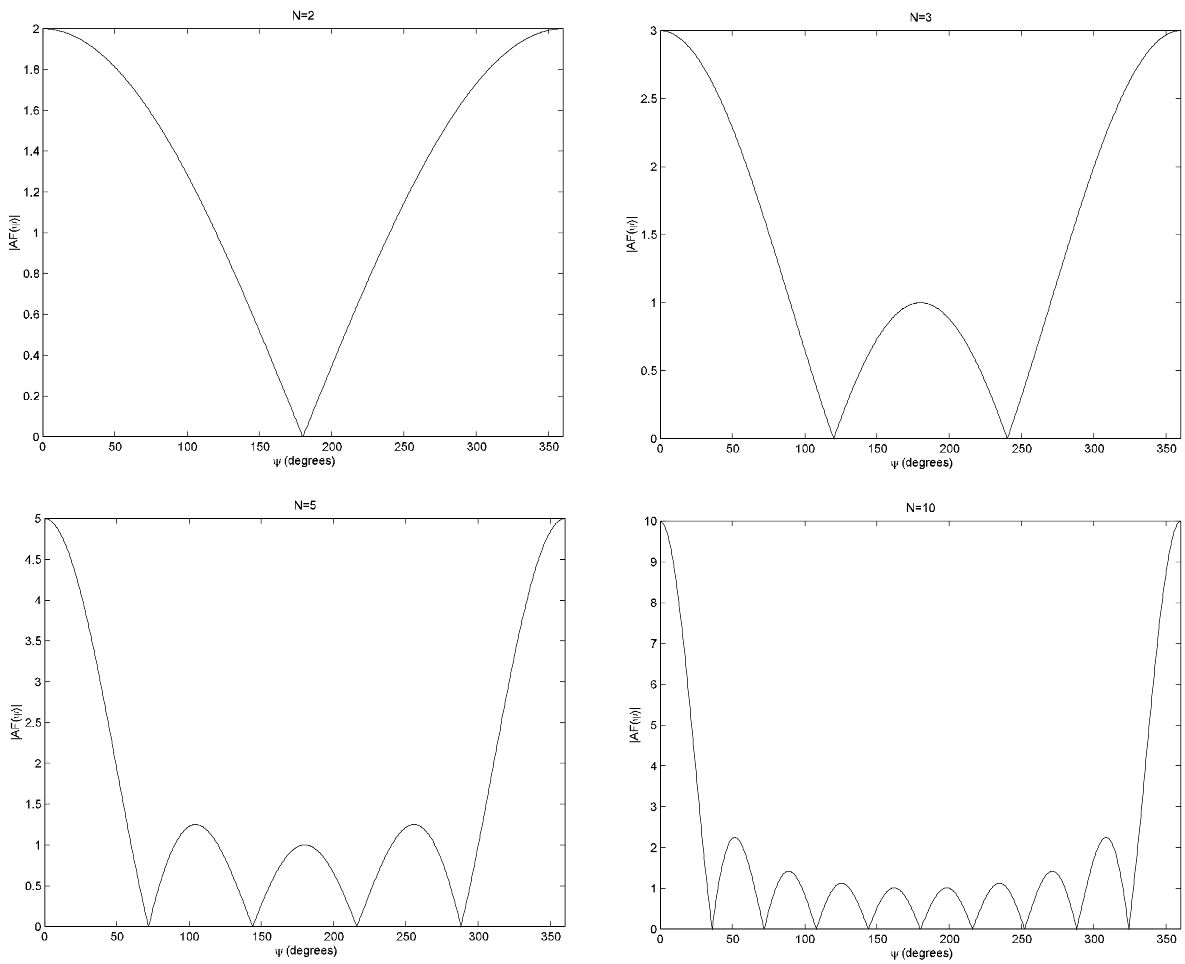
$$AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{e^{jN\frac{\psi}{2}}}{e^{j\frac{\psi}{2}}} \, \frac{e^{jN\frac{\psi}{2}} - e^{-jN\frac{\psi}{2}}}{e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}} = e^{j(N-1)\psi/2} \, \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$$

The complex exponential term in the last expression of the above equation represents the phase shift of the array phase center relative to the origin. If the position of the array is shifted so that the center of the array is located at the origin, this phase term goes away.

The array factor then becomes

$$AF = \frac{\sin\left(\dfrac{N\psi}{2}\right)}{\sin\left(\dfrac{\psi}{2}\right)}$$

Below are plots of the array factor *AF* vs. the array phase function ψ as the number of elements in the array is increased. **Note that these are not plots of *AF* vs. the elevation angle θ.**



Some general characteristics of the array factor *AF* with respect to ψ:

(1) $[AF]_{max} = N$ at $\psi = 0$ (main lobe).

(2) Total number of lobes = N − 1 (one main lobe, N − 2 sidelobes).

(3) Main lobe width = $4\pi/N$, minor lobe width = $2\pi/N$

The array factor may be normalized so that the maximum value for any value of N is unity. The normalized array factor is

$$(AF)_n = \frac{1}{N} \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$$

The nulls of the array function are found by determining the zeros of the numerator term where the denominator is not simultaneously zero.

$$\sin\left(\frac{N\psi}{2}\right) = 0 \quad \Rightarrow \quad \frac{N\psi}{2} = \pm n\pi \quad \Rightarrow \quad \alpha + kd\cos\theta_n = \pm\frac{2n\pi}{N}$$

$$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\alpha \pm \frac{2n\pi}{N}\right)\right] \qquad \begin{array}{c} n = 1,\ 2,\ 3,\ \ldots \\ n \neq 0,\ N,\ 2N,\ 3N,\ \ldots \end{array}$$

The peaks of the array function are found by determining the zeros of the numerator term where the denominator is simultaneously zero.

$$\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-\alpha \pm 2m\pi)\right] \qquad m = 0,\ 1,\ 2,\ \ldots$$

The $m = 0$ term,

$$\theta_m = \cos^{-1}\left(\frac{\lambda\alpha}{2\pi d}\right)$$

represents the angle which makes $\psi = 0$ (main lobe).

Broadside and End-fire Arrays

The phasing of the uniform linear array elements may be chosen such that the main lobe of the array pattern lies along the array axis (*end-fire array*) or normal to the array axis (*broadside array*).

End-fire array	main lobe at $\theta = 0^\circ$ or $\theta = 180^\circ$
Broadside array	main lobe at $\theta = 90^\circ$

The maximum of the array factor occurs when the array phase function is zero.

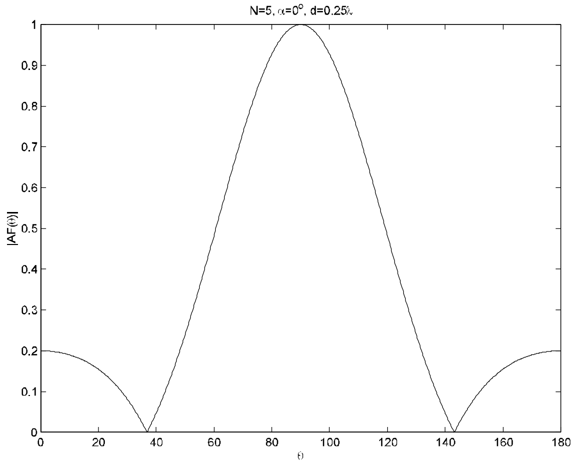
$$\psi = \alpha + kd\cos\theta = 0$$

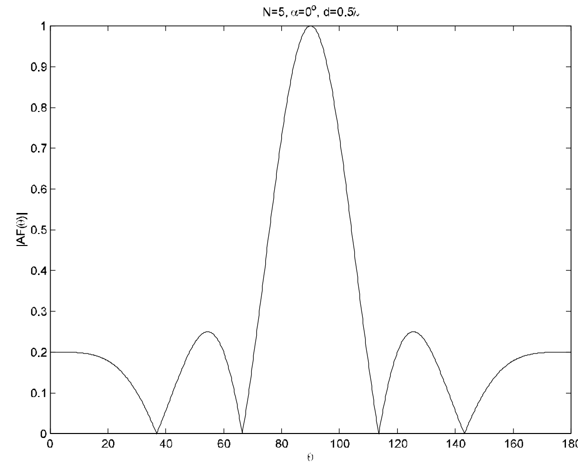
For a broadside array, in order for the above equation to be satisfied with $\theta = 90^\circ$, the phase angle α must be zero. In other words, all elements of the array must be driven with the same phase. With $\alpha = 0^\circ$, the normalized array factor reduces to

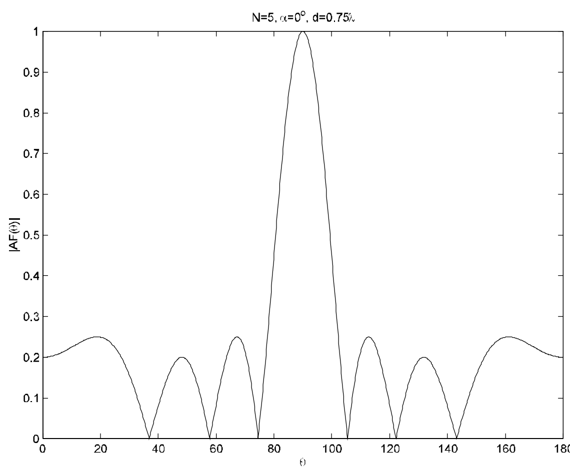
$$(AF)_n = \frac{1}{N}\frac{\sin\left(\frac{Nkd}{2}\cos\theta\right)}{\sin\left(\frac{kd}{2}\cos\theta\right)}$$

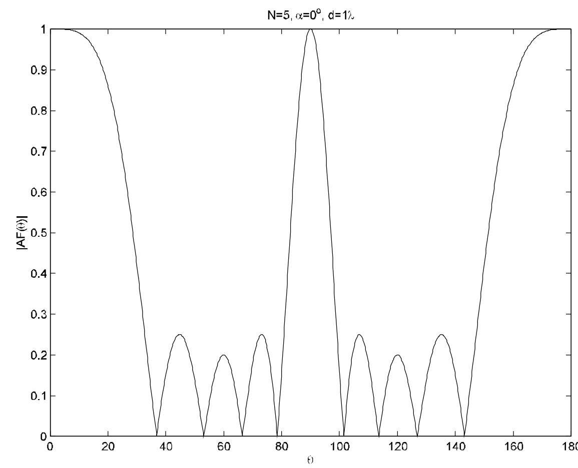
Normalized array function
Broadside array, $\alpha = 0^\circ$

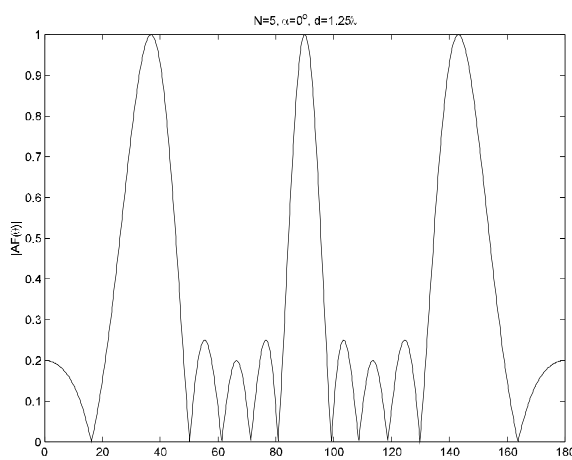
Consider a 5-element broadside array ($\alpha = 0^\circ$) as the element spacing is varied. In general, as the element spacing is increased, the main lobe beamwidth is decreased. However, *grating lobes* (maxima in directions other than the main lobe direction) are introduced when the element spacing is greater than or equal to one wavelength. If the array pattern design requires that no grating lobes be present, then the array element spacing should be chosen to be less than one wavelength.

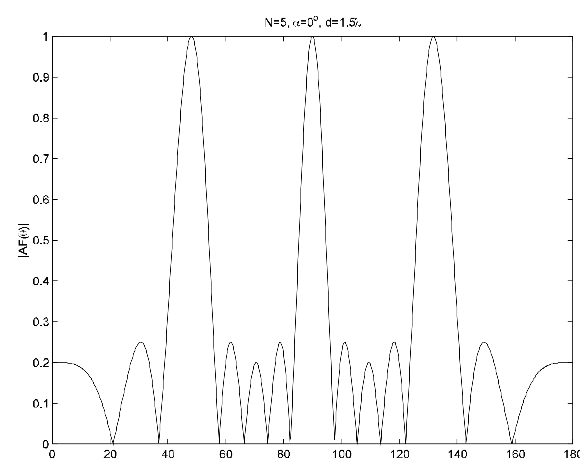

N=5, α=0°, d=0.25λ
|AF(θ)|
θ


N=5, α=0°, d=0.5λ
|AF(θ)|
θ

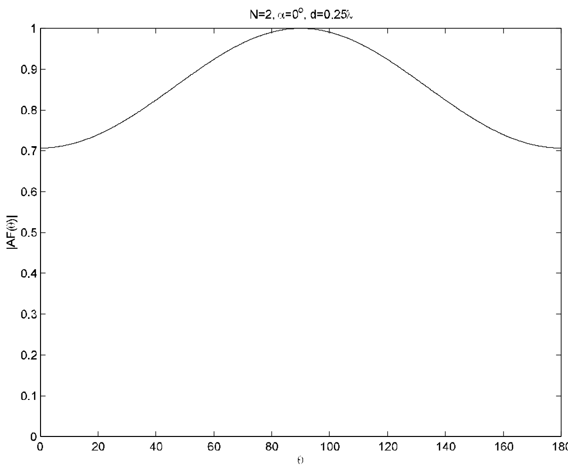

N=5, α=0°, d=0.75λ
|AF(θ)|
θ

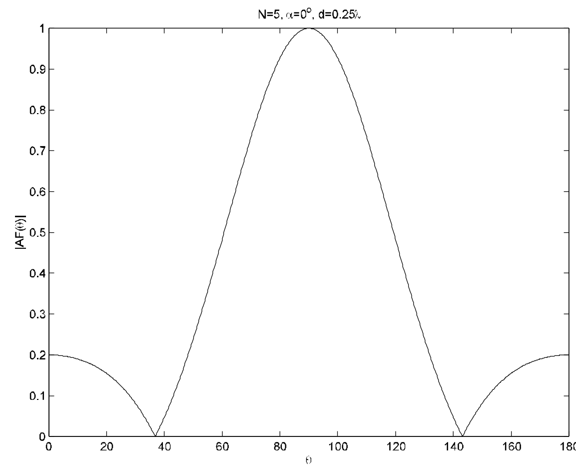

N=5, α=0°, d=1λ
|AF(θ)|
θ

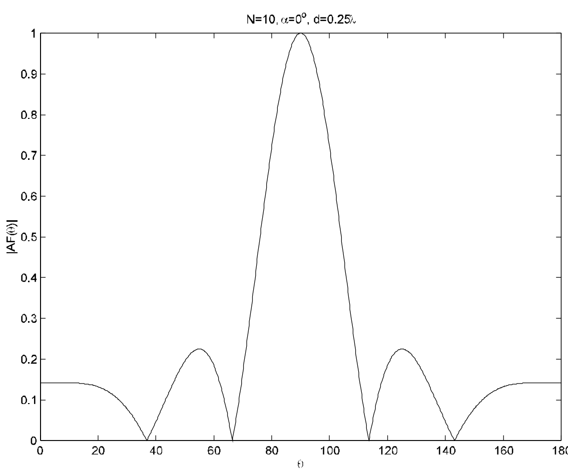

N=5, α=0°, d=1.25λ
|AF(θ)|
θ

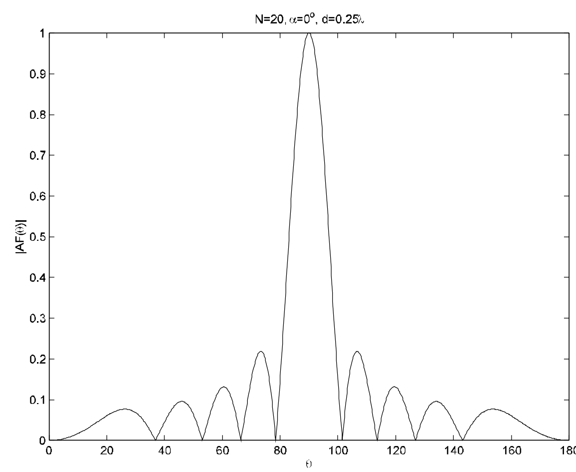

N=5, α=0°, d=1.5λ
|AF(θ)|
θ

If we consider the broadside array factor as a function of the number of array elements, we find that, in general, the main beam is sharpened as the number of elements increases. Below are plots of *AF* for a broadside array ($\alpha = 0^o$) with elements separated by $d = 0.25\lambda$ for $N = 2$, 5, 10 and 20.

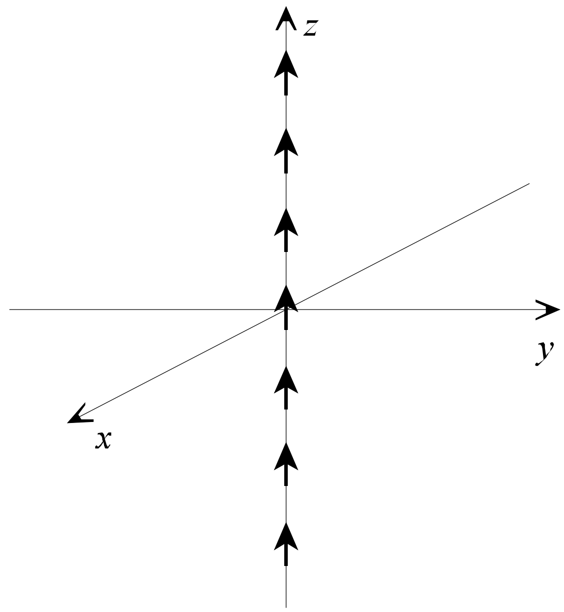








Using the pattern multiplication theorem, the overall array pattern is obtained by multiplying the element pattern by the array factor. As an example, consider an broadside array ($\alpha = 0^o$) of seven short vertical dipoles spaced 0.5λ apart along the z-axis.



The normalized element field pattern for the infinitesimal dipole is

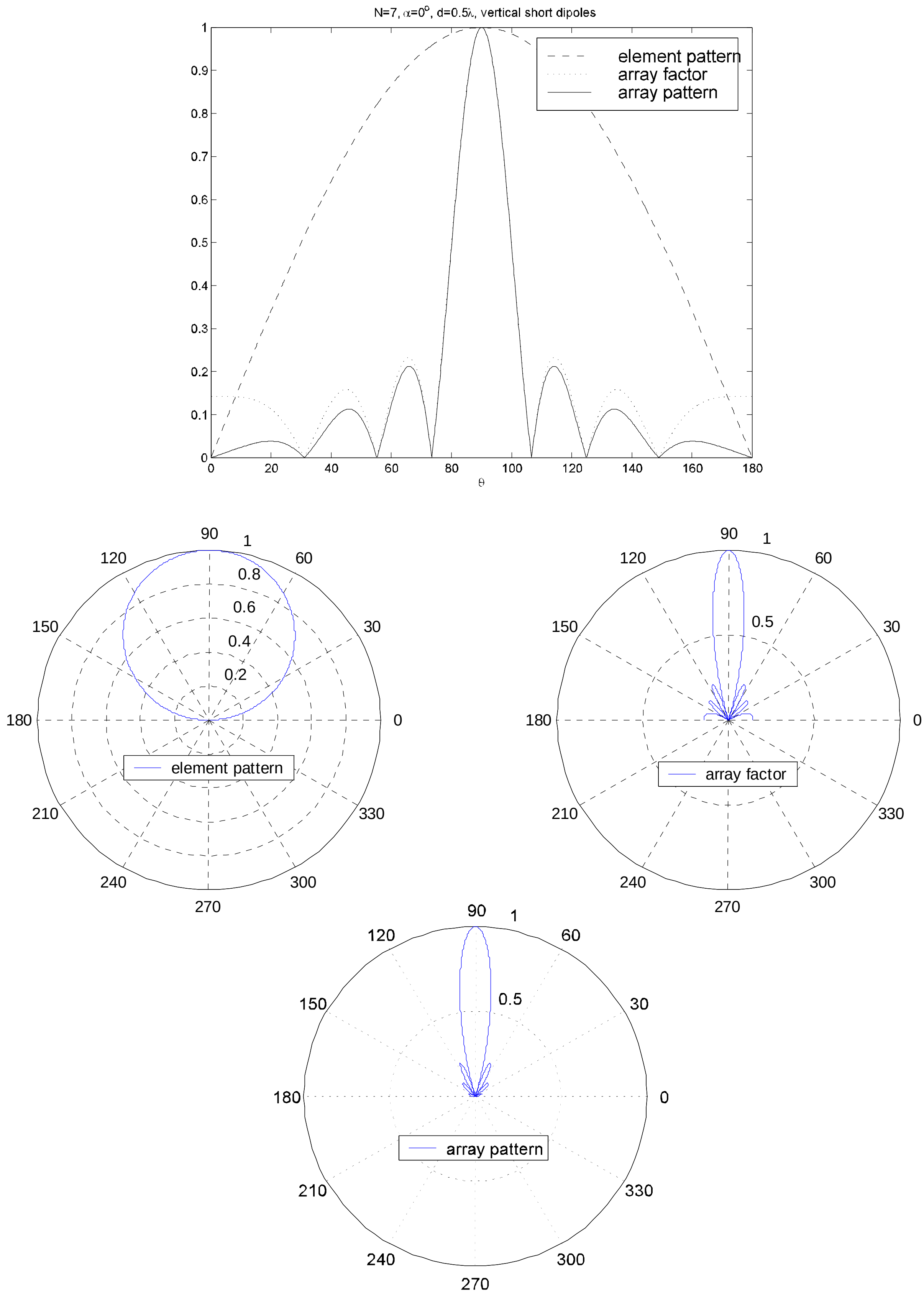
$$f(\theta,\phi) = \sin\theta$$

The array factor for the seven element array is

$$(AF)_n = \frac{1}{7}\,\frac{\sin\left(\frac{7\pi}{2}\cos\theta\right)}{\sin\left(\frac{\pi}{2}\cos\theta\right)}$$

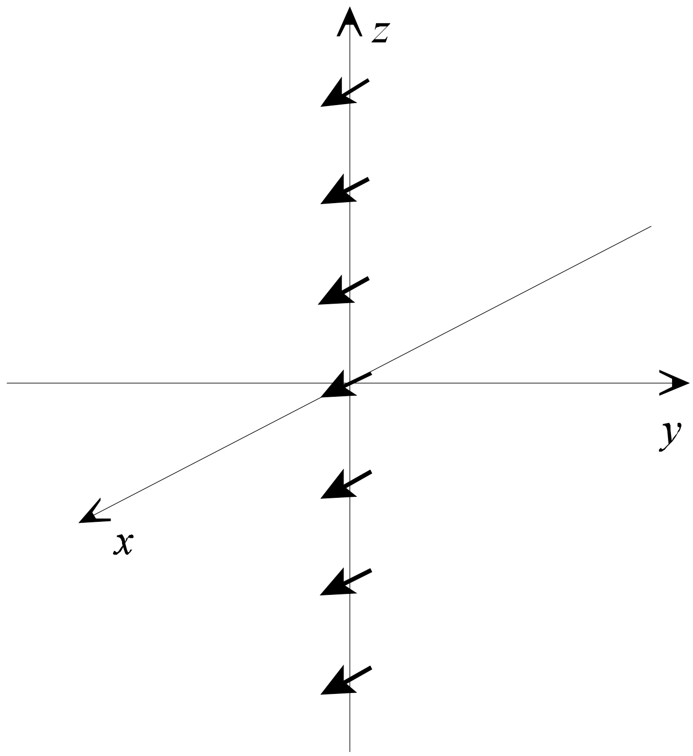
The overall normalized array pattern is

$$F(\theta,\phi) = \frac{1}{7}\,\frac{\sin\left(\frac{7\pi}{2}\cos\theta\right)}{\sin\left(\frac{\pi}{2}\cos\theta\right)}\sin\theta$$


N=7, α=0°, d=0.5λ, vertical short dipoles
element pattern
array factor
array pattern
θ
element pattern
array factor
array pattern

If we consider the same array with horizontal (x-directed) short dipoles, the resulting normalized element field pattern is

$$f(\theta,\phi) = \sqrt{1 - \sin^2\theta\cos^2\phi}$$

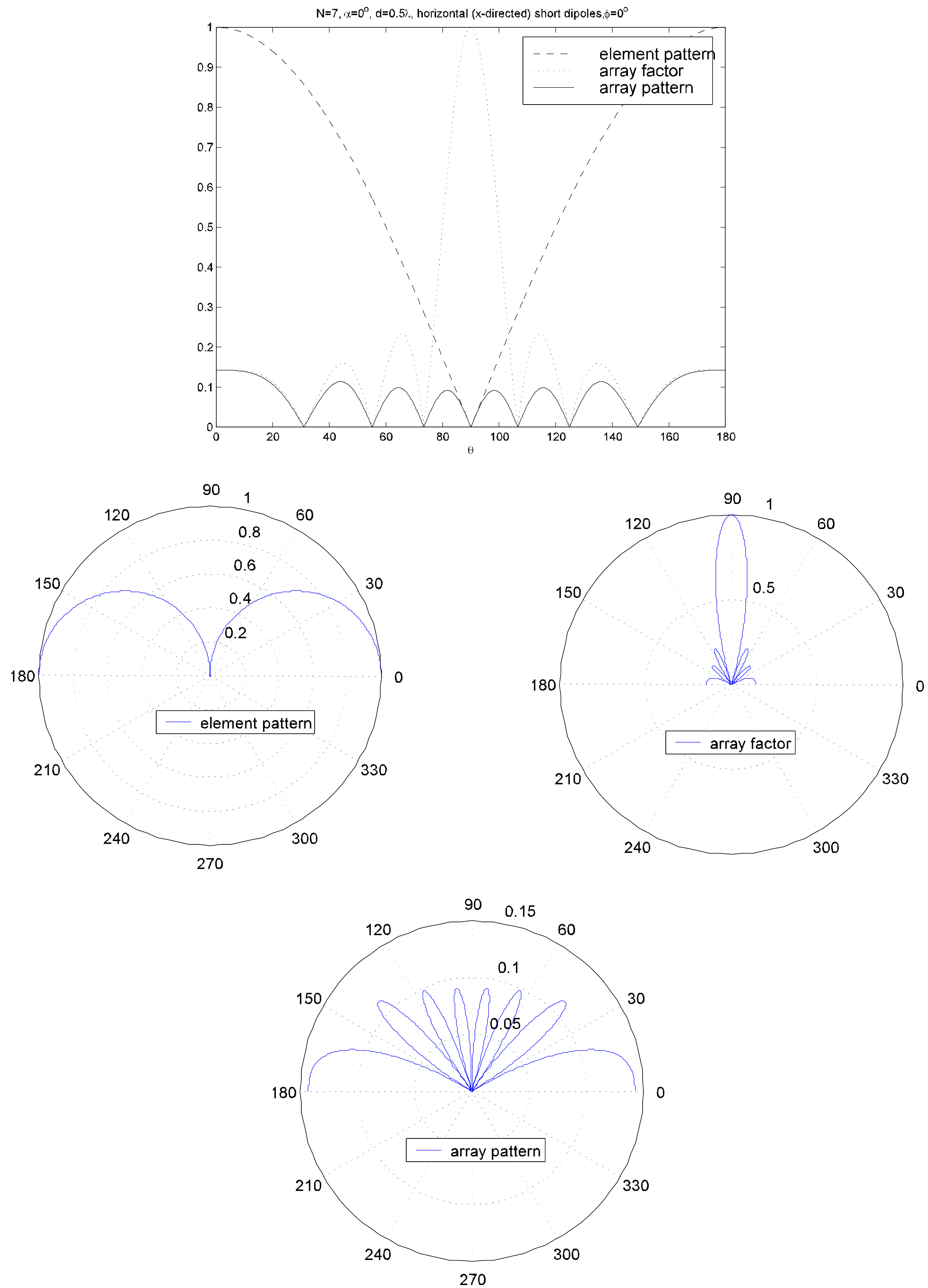


Since the element pattern depends on the angle ϕ, we must choose a value of ϕ to plot the pattern. If we choose $\phi = 0^o$, the element pattern becomes

$$f(\theta,\phi = 0^o) = \sqrt{1 - \sin^2\theta} = \cos\theta$$

and the array pattern is given by

$$F(\theta,\phi) = \frac{1}{7}\frac{\sin\left(\frac{7\pi}{2}\cos\theta\right)}{\sin\left(\frac{\pi}{2}\cos\theta\right)}\cos\theta$$



N=7, α=0°, d=0.5λ, horizontal (x-directed) short dipoles,φ=0°
element pattern
array factor
array pattern
θ
element pattern
array factor
array pattern

If we plot the array pattern for ϕ = 90°, we find that the element pattern is unity and the array pattern is the same as the array factor. Thus, the main beam of the array of *x*-directed short dipoles lies along the *y*-axis. The nulls of the array element pattern along the *x*-axis prevent the array from radiating efficiently in that broadside direction. End-fire arrays may be designed to focus the main beam of the array factor along the array axis in either the θ=0° or θ=180° directions. Given that the maximum of the array factor occurs when

$$\psi = \alpha + kd\cos\theta = 0$$

in order for the above equation to be satisfied with θ = 0°, the phase angle α must be

$$\alpha = -kd$$

For θ = 180°, the phase angle α must be

$$\alpha = +kd$$

which gives

$$\psi = kd(\cos\theta \mp 1) \qquad \theta = \begin{Bmatrix} 0^o \\ 180^o \end{Bmatrix}$$

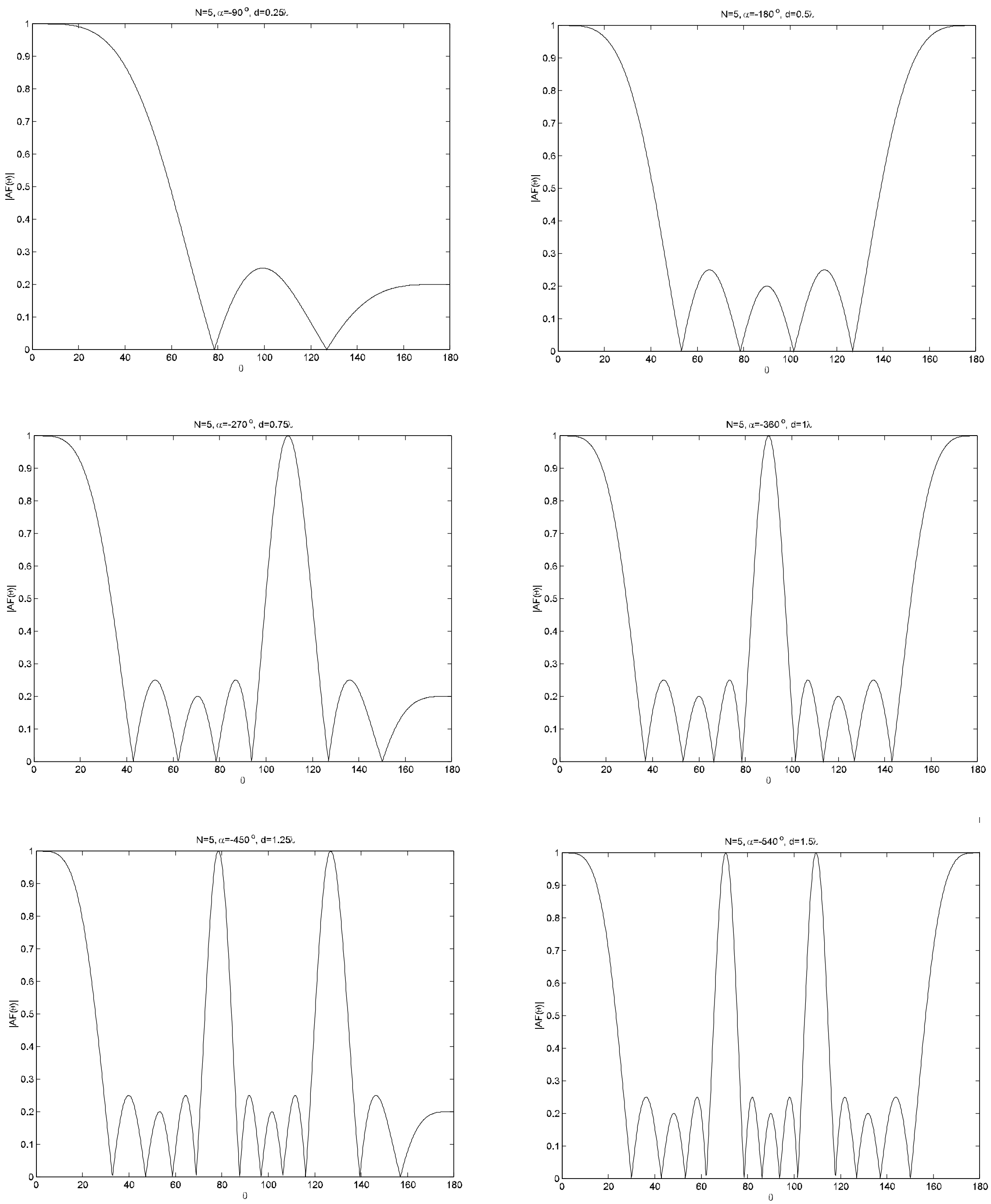
The normalized array factor for an end-fire array reduces to

$$(AF)_n = \frac{1}{N}\,\frac{\sin\left[\frac{Nkd}{2}(\cos\theta \mp 1)\right]}{\sin\left[\frac{kd}{2}(\cos\theta \mp 1)\right]}$$

End-fire array, $\alpha = \mp kd$

Normalized array function

Consider a 5-element end-fire array (θ = 0°) as the element spacing is varied. Note that the phase angle α must change as the spacing changes in order to keep the main beam of the array function in the same direction.



If the corresponding positive phase angles are chosen, the array factor plots are mirror images of the above plots (about $\theta = 90^o$). Note that the end-fire array grating lobes are introduced for element spacings of $d \geq 0.5\lambda$.

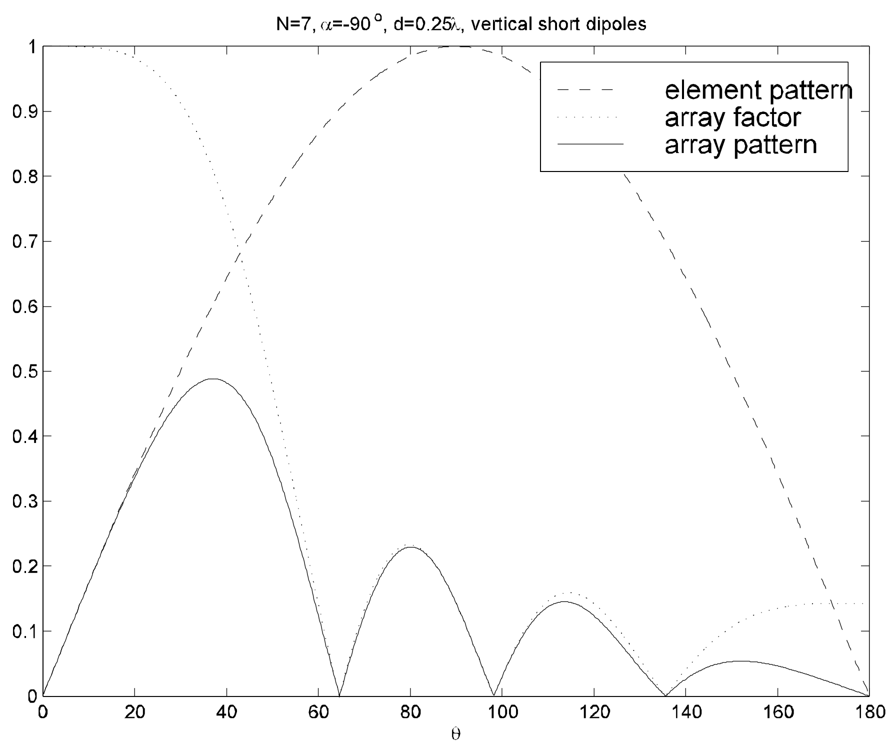
7-element array end-fire array, vertical short dipoles ($d = 0.25\lambda$, $\alpha = -90^o$)

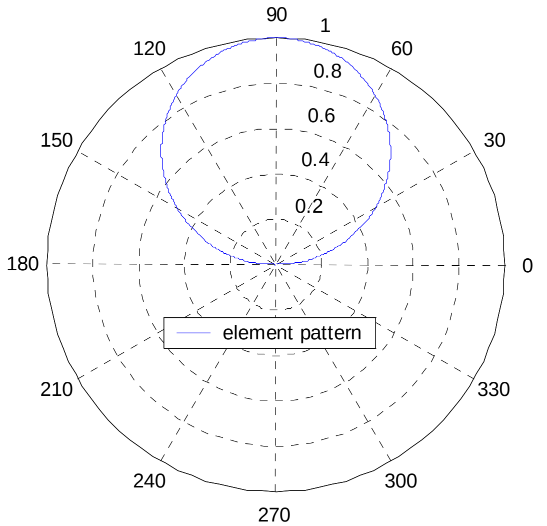
The normalized array factor for the 7-element end-fire array is

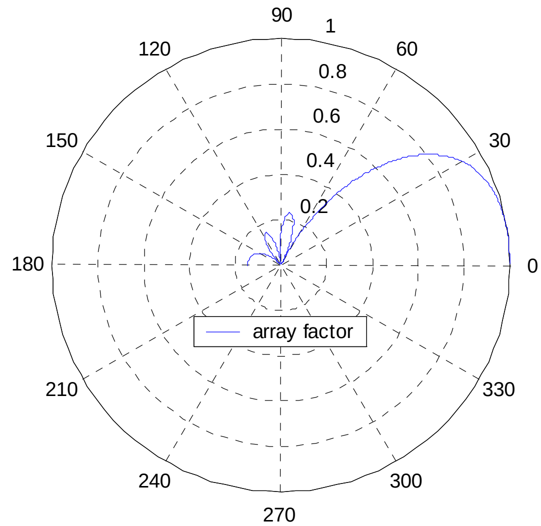
$$(AF)_n = \frac{1}{7}\,\frac{\sin\left[\dfrac{7\pi}{4}(\cos\theta - 1)\right]}{\sin\left[\dfrac{\pi}{4}(\cos\theta - 1)\right]}$$

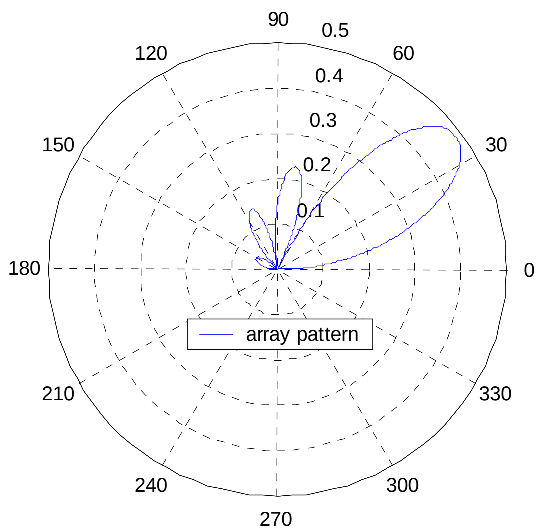
The overall array field pattern is

$$F(\theta,\phi) = \frac{1}{7}\,\frac{\sin\left[\dfrac{7\pi}{4}(\cos\theta - 1)\right]}{\sin\left[\dfrac{\pi}{4}(\cos\theta - 1)\right]}\sin\theta$$





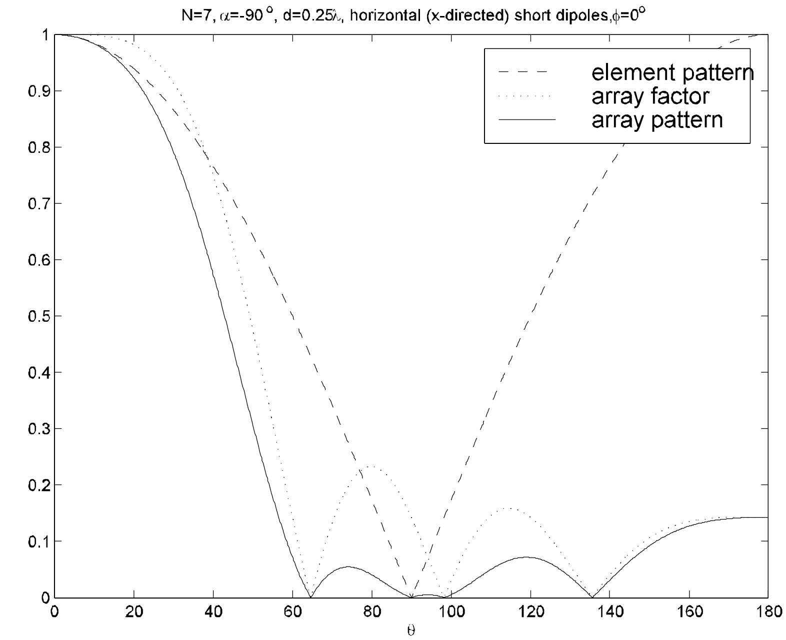


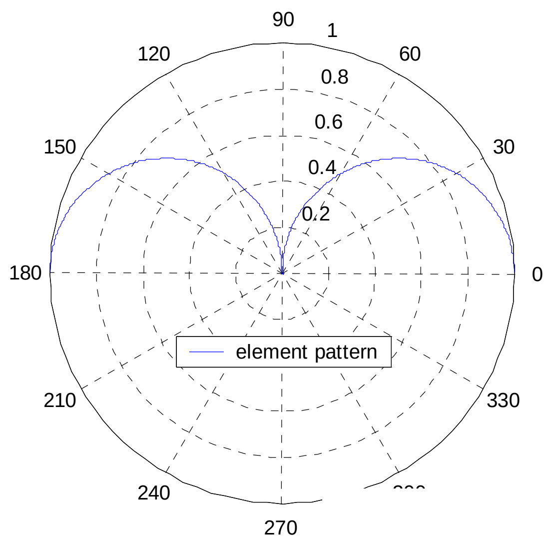


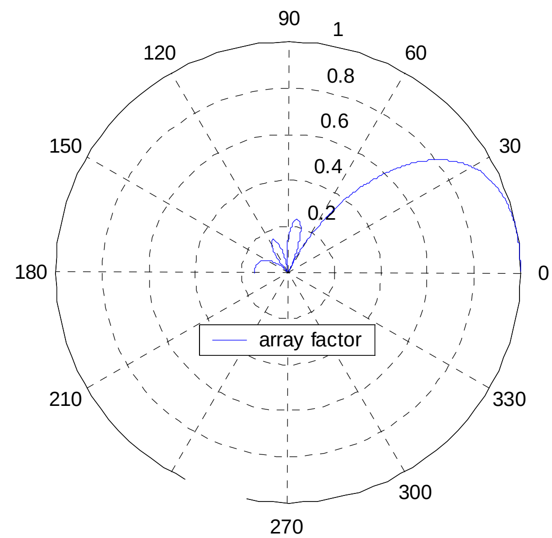
<u>7-element end-fire array, *x*-directed horizontal short dipoles</u>
($d = 0.25\lambda$, $\alpha = -90^{o}$)

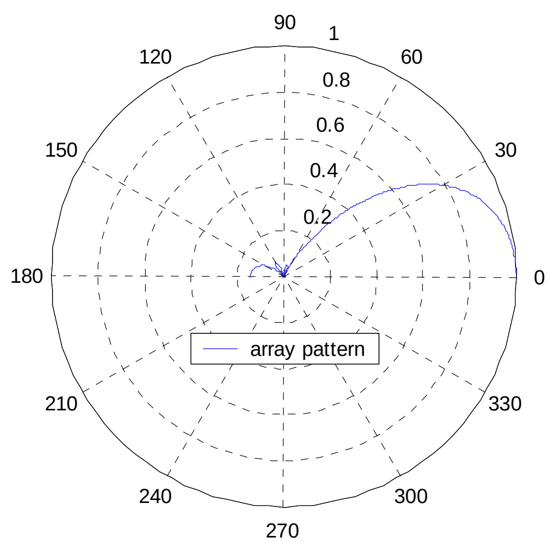
The overall array pattern in the $\phi = 0^{o}$ plane is

$$F(\theta,\phi) = \frac{1}{7}\,\frac{\sin\left[\frac{7\pi}{4}(\cos\theta - 1)\right]}{\sin\left[\frac{\pi}{4}(\cos\theta - 1)\right]}\cos\theta$$


N=7, α=-90°, d=0.25λ, horizontal (x-directed) short dipoles,φ=0°
element pattern
array factor
array pattern
θ


element pattern


array factor


array pattern

Hansen-Woodyard End-fire Array

The Hansen-Woodyard end-fire array is a special array designed for maximum directivity.

Ordinary end-fire array ⇒ $\alpha = \pm kd$

Hansen-Woodyard end-fire array ⇒ $\alpha = \pm\,(kd + \delta)$

In order to increase the directivity in a closely-spaced electrically long end-fire array, Hansen and Woodyard analyzed the patterns and found that a additional phase shift of

$$\delta = \frac{2.94}{N-1} \approx \frac{\pi}{N}$$

increased the directivity of the array over that of the ordinary end-fire array given an element spacing of

$$d = \frac{\lambda}{4}\left(1 - \frac{1}{N}\right)$$

For very long arrays (N - large), the element spacing in the Hansen-Woodyard end-fire array approaches one-quarter wavelength. The Hansen-Woodyard design shown here does not necessarily produce the maximum directivity for a given linear array but does produce a directivity larger than that of the ordinary end-fire array [by a factor of approximately 1.79 (2.5 dB)]. The Hansen-Woodyard end-fire array design can be summarized as

$$\alpha = \mp\left(kd + \frac{\pi}{N}\right) \qquad d = \frac{\lambda}{4}\left(1 - \frac{1}{N}\right) \qquad \text{(Hansen-Woodyard end-fire array)}$$

where the upper sign produces a maximum in the $\theta = 0^{\circ}$ direction and the lower sign produces a maximum in the $\theta = 180^{\circ}$ direction. The Hansen-Woodyard end-fire design increases the directivity of the array at the expense of higher sidelobe levels.

Non-Uniformly Excited, Equally-Spaced Arrays

Given a two element array with equal current amplitudes and spacing, the array factor is

$$AF = 1 + e^{j\psi}$$

For a broadside array ($\alpha = 0^o$) with element spacing d less than one-half wavelength, the array factor has no sidelobes. An array formed by taking the product of two arrays of this type gives

$$AF = (1 + e^{j\psi})(1 + e^{j\psi}) = 1 + 2e^{j\psi} + e^{j2\psi}$$

This array factor, being the square of an array factor with no sidelobes, also has no sidelobes. Mathematically, the array factor above represents a 3-element equally-spaced array driven by current amplitudes with ratios of 1:2:1. In a similar fashion, equivalent arrays with more elements may be formed.

$$\begin{array}{ll} 2\text{-element} & AF = (1 + e^{j\psi}) \\ 3\text{-element} & AF = (1 + e^{j\psi})^2 = 1 + 2e^{j\psi} + e^{j2\psi} \\ 4\text{-element} & AF = (1 + e^{j\psi})^3 = 1 + 3e^{j\psi} + 3e^{j2\psi} + e^{j3\psi} \\ \vdots & \\ N\text{-element} & AF = (1 + e^{j\psi})^{(N-1)} \end{array}$$

The current coefficients of the resulting N-element array take the form of a binomial series. The array is known as a *binomial array*.

$$\begin{aligned} AF &= (1 + e^{j\psi})^{(N-1)} \\ &= 1 + (N-1)e^{j\psi} + \frac{(N-1)(N-2)}{2!}e^{j2\psi} \\ &\quad + \frac{(N-1)(N-2)(N-3)}{3!}e^{j3\psi} + \ldots \end{aligned}$$

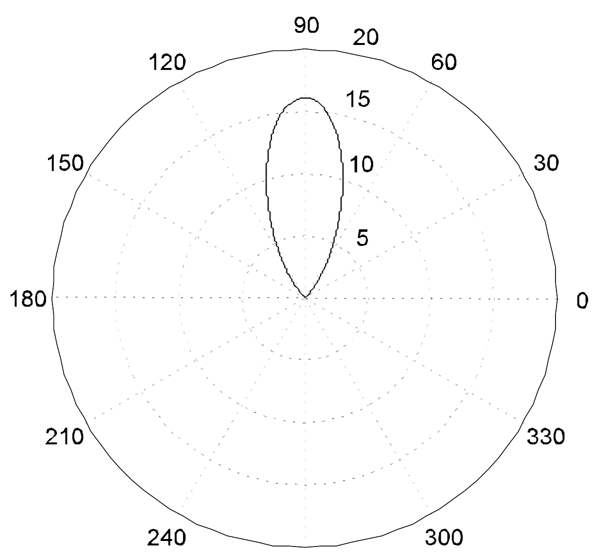
Binomial array

The excitation coefficients for the binomial array are given by Pascal's triangle.

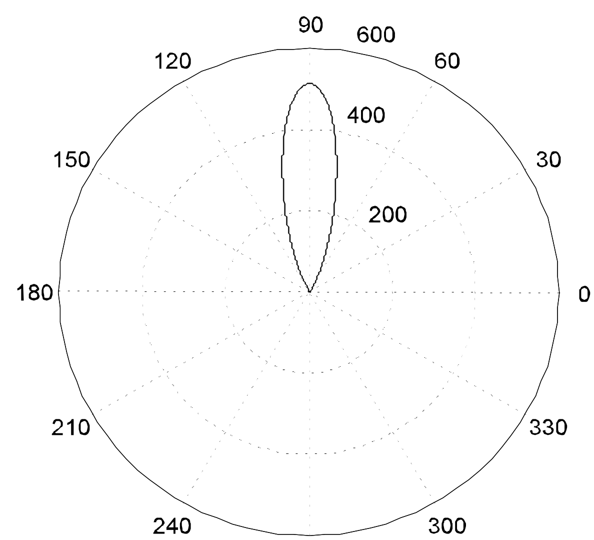
$$
\begin{array}{cl}
1 & N=1 \\
1 \quad 1 & N=2 \\
1 \quad 2 \quad 1 & N=3 \\
1 \quad 3 \quad 3 \quad 1 & N=4 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 & N=5 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 & N=6 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 & N=7 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 & N=8 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 & N=9 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 & N=10
\end{array}
$$

The binomial array has the special property that the array factor has no sidelobes for element spacings of $\lambda/2$ or less. Sidelobes are introduced for element spacings larger than $\lambda/2$.

$N = 5$, $d = 0.5\lambda$



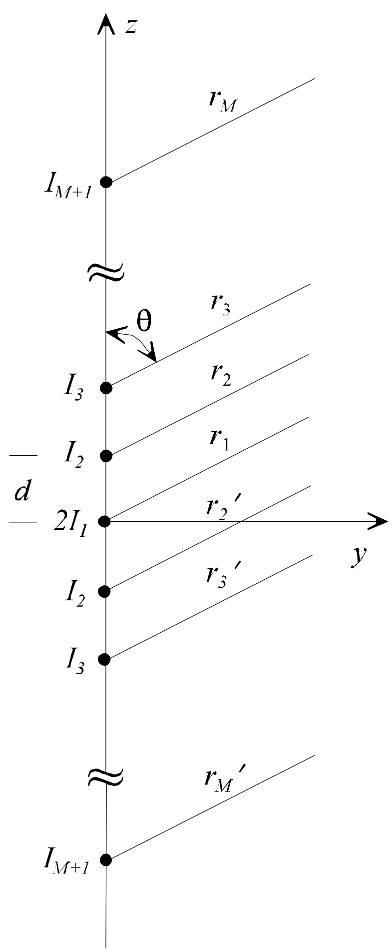
$N = 10$, $d = 0.5\lambda$



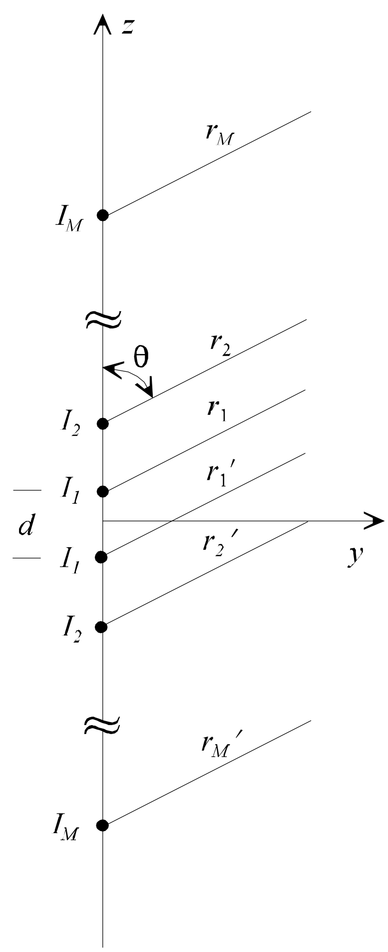
Array Factor - Uniform Spacing, Nonuniform Amplitude

Consider an array of isotropic elements positioned symmetrically along the *z*-axis (total number of elements = *P*). The array factor for this array will be determined assuming that all elements are excited with the same current phase ($\phi = 0^o$ for simplicity) but nonuniform current amplitudes. The amplitude distribution assumed to be symmetric about the origin.

$P = 2M + 1$ (Odd)



$P = 2M$ (Even)



P = 2M + 1 (Odd)

$$E_{M+1} \sim I_{M+1}\,\frac{e^{-jk(r - Md\cos\theta)}}{4\pi r}$$

$$= a_{M+1} I_o E_o e^{jkMd\cos\theta}$$

$$\vdots$$

$$E_3 \sim I_3\,\frac{e^{-jk(r - 2d\cos\theta)}}{4\pi r}$$

$$= a_3 I_o E_o e^{jk2d\cos\theta}$$

$$E_2 \sim I_2\,\frac{e^{-jk(r - d\cos\theta)}}{4\pi r}$$

$$= a_2 I_o E_o e^{jkd\cos\theta}$$

$$E_1 \sim 2I_1\,\frac{e^{-jkr}}{4\pi r} = 2a_1 I_o E_o$$

$$E_2' \sim I_2\,\frac{e^{-jk(r + d\cos\theta)}}{4\pi r}$$

$$= a_2 I_o E_o e^{-jkd\cos\theta}$$

$$E_3' \sim I_3\,\frac{e^{-jk(r + 2d\cos\theta)}}{4\pi r}$$

$$= a_3 I_o E_o e^{-jk2d\cos\theta}$$

$$\vdots$$

$$E_{M+1}' \sim I_{M+1}\,\frac{e^{-jk(r + Md\cos\theta)}}{4\pi r}$$

$$= a_{M+1} I_o E_o e^{-jkMd\cos\theta}$$

P = 2M (Even)

$$E_M \sim I_M\,\frac{e^{-jk(r - \frac{2M-1}{2}d\cos\theta)}}{4\pi r}$$

$$= a_M I_o E_o e^{jk\frac{2M-1}{2}d\cos\theta}$$

$$\vdots$$

$$E_2 \sim I_2\,\frac{e^{-jk(r - \frac{3}{2}d\cos\theta)}}{4\pi r}$$

$$= a_2 I_o E_o e^{jk\frac{3}{2}d\cos\theta}$$

$$E_1 \sim I_1\,\frac{e^{-jk(r - \frac{1}{2}d\cos\theta)}}{4\pi r}$$

$$= a_1 I_o E_o e^{jk\frac{1}{2}d\cos\theta}$$

$$E_1' \sim I_1\,\frac{e^{-jk(r + \frac{1}{2}d\cos\theta)}}{4\pi r}$$

$$= a_1 I_o E_o e^{-jk\frac{1}{2}d\cos\theta}$$

$$E_2' \sim a_2 I_2\,\frac{e^{-jk(r + \frac{3}{2}d\cos\theta)}}{4\pi r}$$

$$= I_o E_o e^{-jk\frac{3}{2}d\cos\theta}$$

$$\vdots$$

$$E_M' \sim I_M\,\frac{e^{-jk(r + \frac{2M-1}{2}d\cos\theta)}}{4\pi r}$$

$$= a_M I_1 E_o e^{-jk\frac{2M-1}{2}d\cos\theta}$$

$\underline{P = 2M + 1 \text{ (Odd)}}$

$$(E)_P = E_{M+1} + \cdots + E_3 + E_2 + E_1 + E_2^{/} + E_3^{/} + \cdots E_{M+1}$$
$$= 2I_o E_o \{ a_1 + a_2 \cos(kd\cos\theta) + a_3 \cos(2kd\cos\theta) + \cdots + a_{M+1} \cos(Mkd\cos\theta) \}$$

$$(AF)_P = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)\frac{\pi d}{\lambda}\cos\theta\right] = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where $u = \dfrac{\pi d}{\lambda}\cos\theta$

$\underline{P = 2M \text{ (Even)}}$

$$(E)_P = E_M + \cdots + E_2 + E_1 + E_1^{/} + E_2^{/} + \cdots + E_M$$
$$= 2I_o E_o \left\{ a_1 \cos\left(\frac{1}{2}kd\cos\theta\right) + a_2 \cos\left(\frac{3}{2}kd\cos\theta\right) + \cdots + a_M \cos\left(\frac{2M-1}{2}kd\cos\theta\right) \right\}$$

$$(AF)_P = \sum_{n=1}^{M} a_n \cos\left[(2n-1)\frac{\pi d}{\lambda}\cos\theta\right] = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

Note that the array factors are coefficients multiplied by cosines with arguments that are integer multiples of u. Using trigonometric identities, these cosine functions can be written as powers of u.

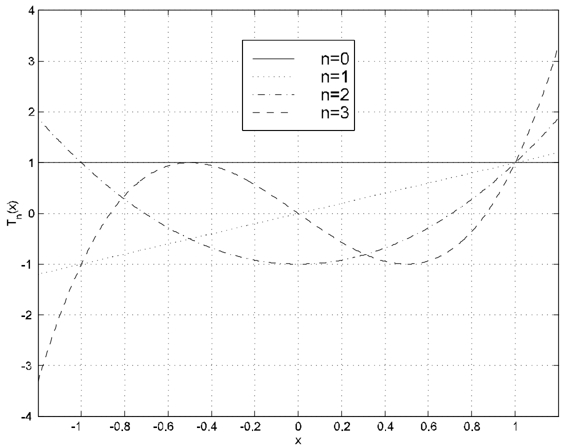
$$
\begin{aligned}
\cos 0u &= 1 \\
\cos 1u &= \cos u \\
\cos 2u &= 2\cos^2 u - 1 \\
\cos 3u &= 4\cos^3 u - 3\cos u \\
\cos 4u &= 8\cos^4 u - 8\cos^2 u + 1 \\
\cos 5u &= 16\cos^5 u - 20\cos^3 u + 5\cos u \\
\cos 6u &= 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1 \\
\cos 7u &= 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u \\
\cos 8u &= 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1 \\
\cos 9u &= 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u \\
&\vdots
\end{aligned}
$$

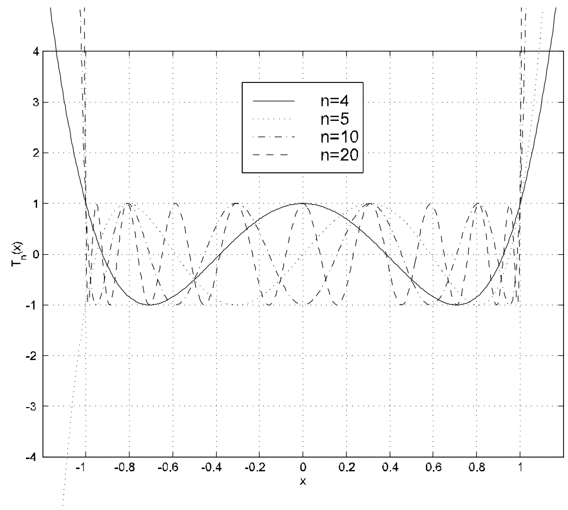
Through the transformation of $x = \cos u$, the terms may be written as a set of polynomials [Chebyshev polynomials - T_n(x)].

$$
\begin{aligned}
\cos 0u &= 1 = T_0(x) \\
\cos 1u &= x = T_1(x) \\
\cos 2u &= 2x^2 - 1 = T_2(x) \\
\cos 3u &= 4x^3 - 3x = T_3(x) \\
\cos 4u &= 8x^4 - 8x^2 + 1 = T_4(x) \\
\cos 5u &= 16x^5 - 20x^3 + 5x = T_5(x) \\
\cos 6u &= 32x^6 - 48x^4 + 18x^2 - 1 = T_6(x) \\
\cos 7u &= 64x^7 - 112x^5 + 56x^3 - 7x = T_7(x) \\
\cos 8u &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 = T_8(x) \\
\cos 9u &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x = T_9(x) \\
&\vdots
\end{aligned}
$$

Using properties of the Chebyshev polynomials, we may design arrays with specific sidelobe characteristics. Namely, we may design arrays with all sidelobes at some prescribed level.

Chebyshev Polynomials





Properties of Chebyshev Polynomials

1. Even ordered Chebyshev polynomials are even functions.
2. Odd ordered Chebyshev polynomials are odd functions.
3. The magnitude of any Chebyshev polynomial is unity or less in the range of $-1 \leq x \leq 1$.
4. $T_n(1) = 1$ for all Chebyshev polynomials.
5. All zeros (roots) of the Chebshev polynomials lie within the range of $-1 \leq x \leq 1$.

Using the properties of Chebyshev polynomials, we may design arrays with all sidelobes at a prescribed level below the main beam (Dolph-Chebyshev array). The order of the Chebyshev polynomial should be one less than the total number of elements in the array ($P-1$).

Dolph-Chebyshev Array Design Procedure

(1.) Select the appropriate *AF* for the total number of elements (P).

$$(AF)_P = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right] \qquad P = 2M \quad \text{(even)}$$

$$(AF)_P = \sum_{n=1}^{M} a_n \cos\left[2(n-1)u\right] \qquad P = 2M+1 \quad \text{(odd)}$$

(2.) Replace each cos(mu) term in the array factor by its expansion in terms of powers of cos(u).

(3.) For the required main lobe to side lobe ratio (R_o), find x_o such that

$$R_o = T_{P-1}(x_o) = \cosh\left[(P-1)\cosh^{-1} x_o\right]$$

$$x_o = \cosh\left[\frac{\cosh^{-1} R_o}{P-1}\right]$$

(4.) Substitute cos(u) = x/x_o into the array factor of step 2. This substitution normalizes the array factor sidelobes to a peak of unity.

(5.) Equate the array factor of step 4 to $T_{P-1}(x)$ and determine the array coefficients.

<u>Example</u>

Design a 5-element Dolph-Chebyshev array with $d = 0.5\lambda$ and sidelobes which are 20 dB below the main beam.

(1.) $P = 5$, $M = 2$

$$(AF)_5 = \sum_{n=1}^{3} a_n \cos\left[2(n-1)u\right]$$

(2.) $\cos 0u = 1$

$\cos 2u = 2\cos^2 u - 1$

$\cos 4u = 8\cos^4 u - 8\cos^2 u + 1$

(3.) $$20\,\text{dB} = 20\log_{10} R_o \quad \Rightarrow \quad R_o = 10^1 = 10$$

$$x_o = \cosh\left[\frac{\cosh^{-1} R_o}{P-1}\right] = \cosh\left[\frac{\cosh^{-1} 10}{4}\right] = 1.293$$

(4.) $$(AF)_5 = a_1 + a_2\left[2\left(\frac{x}{x_o}\right)^2 - 1\right] + a_3\left[8\left(\frac{x}{x_o}\right)^4 - 8\left(\frac{x}{x_o}\right)^2 - 1\right] = T_4(x) = 8x^4 - 8x^2 + 1$$

(5.) Equate coefficients and solve for a_1, a_2, and a_3.

$$a_1 = 2.698 \qquad a_2 = 4.493 \qquad a_3 = 2.795$$

$$(AF)_5 = 2.698 + 4.493\cos[\pi\cos\theta] + 2.795\cos[2\pi\cos\theta]$$